Review Final Exam , MTH 205, Fall 2014

Ayman Badawi

QUESTION 1. Solve for $y(x)$: $y^{(2)} - \frac{y'}{x^2} + \frac{y}{x^3} = \frac{10}{x^3}$. Given $y(x) = x$ is a solution to the associated homogenous equation.

QUESTION 2. An object weighing 8 pounds stretches a spring 2feet. Assume that an air-resistance is numerically equals to 2 times the velocity of the motion x(t) acts on the system. a) Determine the equation of motion x(t) if the object is initially released from the equilibrium position with an upward velocity $3ft/s$.

b) Will the spring ever return to the equilibrium position? explain

c) If the answer to (b)is no, then at any time t > 0, will the motion of the spring be above or below the equilibrium position? Explain

QUESTION 3. Solve for $x(t), y(t)$

$x'(t) - y(t) = 0$

$x(t) + y^{(2)}(t) = t^2$, where $x(0) = 0, y(0) = 0, y'(0) = 2$

QUESTION 4. Let $A(t)$ be the population of a small town at time t where t is time in years. Given that the population of the town now is 1000, and the rate of growth is proportional to $(\frac{1}{A(t)} + A(t))$. If the population of the town after 1 year is 1200, what will be the population of the town after 3 years?

QUESTION 5. Given $y' = -y^4 + 9y^2$. Find the critical points of the D.E, and label each as STABLE, SEMI-STABLE, NON-STABLE. If the graph of a solution to the D.E is passing through the point (4, 0), then sketch a rough graph of this solution. If the graph of a solution to the D.E. is passing through the point (4, -2), then sketch a rough graph of this solution.

QUESTION 6. Given $y = xe^x$ is a solution to the D.E : $ay^{(2)} + by' + y = e^x$, where a, b are some constants. Find the general solution to the D.E: $ay^{(2)} + by\prime + y = 0$.

QUESTION 7. Find the solution to $\frac{dy}{dx} = \frac{1}{x+4y^3x^3e^{-2y}}$

QUESTION 8. Solve the D.E : $\frac{dy}{dx} = (x + y)^2 sin^2(\frac{x+y-1}{x+y}) - 1$

QUESTION 9. Find the general solution to $2xy^{(2)} - 10y' + \frac{18}{x}y = 0$. If $y(1) = 10$, and $y'(1) = 31$, what is THE SOLUTION of the D.E.

QUESTION 10. Solve for $y(x)$ such that $\int_0^x xe^{(x-2)}y'(t)\ dt = x^2U(x-1)$, and $y(2) = 12$

QUESTION 11. Find the general solution to $sin(x)y^{(2)} - cos(x)y' = 1$ [hint $\int csc^2(x)\ dx = -cot(x) + c$]

QUESTION 12. Solve for $x(t)$ and $y(t)$: $x'(t) - y(t) = 0$, $x(t) + \int_0^t y(r)\ dr = 2t$, $x(1) = 1$.

QUESTION 13. (i) $y' = \frac{x^2+y^2}{xy}$ [Hint: by rearranging the equation we have $y' - \frac{y}{x} = \frac{x}{y}$, Bernouli, here $n = -1, 1-n = 2$, thus $w = y^2$, the solution is $y^2 = x^2ln(x^2) + cx^2 = 2x^2ln(|x|) + cx^2$]

(ii) $y' = (2x + x^2)e^{(x+3y)}$ [Hint: separable, $y' = \frac{(2x+x^2)e^x}{e^{-3y}}$, solution $\frac{-1}{3}e^{-3y} = x^2e^x + c$]

(iii) $y' = \frac{x^2+2}{y}$ [Hint: separable, $\int y\ dy = \int (x^2 + 2)\ dx$, do it]

(iv) $y' = (x + y)^2 + 8$ [Hint: reduced to sparable $w = (x + y)$, $dw/dx = 1 + dy/dx$, $dw/dx = w^2 + 9$, solution: $3tan^{-1}(w/3) = x + c$, hence $3tan^{-1}((x + y)/3) = x + c$]

QUESTION 14. (i) $e^xy' + \frac{e^x}{x}y = 1$ and $x > 0$[You end up with $y = \frac{\int xe^{-x}\ dx}{x}$ so finish it now]

(ii) $x^2y' + y = 1$ [you end up with $y = -x\int \frac{e^{-1/x}}{x^2}\ dx$]

(iii) $y' + 2y = [2xe^{x^2} + e^{x^2}]\sqrt{y}$ [you end up with $[0.5\frac{\int [2xe^{x^2}+e^{x^2}]e^x}{} dxe^x]$

(iv) $y' + 3xy = x^3\sqrt[3]{y^2}$

QUESTION 15. (i) find $\ell^{-1}\{\frac{3^{-s}}{s(s+2)}\}$

(ii) Find $\ell^{-1}\{\frac{s^3+24}{s^5}\}$

(iii) Find $\ell^{-1}\{\frac{e^{-2s}}{(s+4)^2+4}\}$

(iv) Find $\ell\{u(x-1)e^{(x-1)}sin(x-1)\}$

(v) Find $\ell^{-1}\{\frac{s+2}{s^2+4s+5}\}$

(vi) Find $\ell\{\int_0^x e^{2x-r}sin(r)\,dr\}$

QUESTION 16. Find the largest interval around x so that the LDE: $\frac{x-3}{\sqrt{3x+6}})y^{(4)} + (x-1)y = x^2 + 13, y'(1) = 7, y(1) = -6$ has a unique solution.

QUESTION 17. 1) Show that $c_1 sin(x) + c_2 cos(x)$ is a solution to the LDE : $y^{(2)} + y = 0$, where c_1, c_2 are some constants.
2) Assume that $y'(\pi) = 1$ and $y(\pi/2) = 1$. Show that the given LDE has no solution in this case. Does this contradict the Initial Value Theorem?
3) Assume that $y'(\pi) = -1$ and $y(\pi/2) = 1$. Show that the given LDE has infinitely many solutions. Does this contradict the Initial Value Theorem?

QUESTION 18. 1) Find $\ell\{e^{4x}\}$,

2)$\ell^{-1}\{\frac{s+3}{s^2-7s+6}\}$.

3) Find $\ell\{5^{(2x+1)}\}$

4) Find $\ell^{-1}\{\frac{7-x}{(s+6)^4}\}$

QUESTION 19. Solve the following DE (use Laplace): 1) $y^{(2)} + 8y' + 12y = e^{-2x}$, $y(0) = 0$, $y'(0) = 0$.

2) $2y^{(2)} + 3y' + y = sin(2x)0$, $y(0) = 3$ and $y'(0) = -2$

QUESTION 20. (i) Find $\ell\{U(x-3)7^{2x}\}$

(ii) Find $\ell\{\int_0^x e^{-3r}sin(2r)\,dr\}$ [Hint : Note $e^{-3r} = e^{-3x}e^{3x-3r}$. Thus $\ell\{\int_0^x e^{-3r}sin(2r)\,dr\} = \ell\{e^{-3x}\int_0^x e^{3x-3r}sin(2r)\,dr\}$]

(iii) Find $\ell^{-1}\{\frac{s+10}{(s+4)^4}\}$

(iv) Find $\ell^{-1}\{\frac{s5^{-s}}{(s+3)^2+4}\}$

(v) Use CONVOLUTION Twice to find $\ell^{-1}\{\frac{1}{s^2(s^2+9)}\}$

(vi) Use convolution to find $\ell^{-1}\{\frac{1}{(s+4)^2((s+4)^2+9)}\}$ [Hint: Use 5]

(vii) Find $\ell^{-1}\{\frac{8e^{-3s}}{s^2-4}\}$

(viii) Let

$$k(x) = \begin{cases} 2 & 0 \le x < 4 \\ 6 & x \ge 4 \end{cases}$$

Solve $y^{(2)} - 6y' + 8y = k(x)$, y(0) = $y'(0) = 0$ [hint : first write K(x) in terms of Unit function]

(ix) Solve : $y'(x) - \int_0^x 2e^{x-r}y(r)\,dr = xe^x$, $y(0) = 0$

QUESTION 21. Use undetermined Coeff. Method to solve for $y(x)$:

(i) $y^{(2)} + 6y' - 7y = 0$

(ii) $y^{(6)} - 7y^{(5)} + 10y^{(4)} = 0$

(iii) $y^{(4)} + 6y^{(3)} + 9y^{(2)} = 0$

(iv) $y^{(6)} - 7y^{(5)} + 10y^{(4)} = 10$

(v) $y^{(2)} + 6y' - 7y = e^{-7x}$

(vi) $y^{(2)} + 2y' + 20y = 0$

(vii) $y^{(2)} + 2y' + 5y = 2x + 3$

(viii) $y^{(2)} + 6y' - 7y = cos(x)$

(ix) $y^{(2)} + 2y' + 5y = cos(x)$

(x) $y^{(2)} + 4y = sin(2x)$

QUESTION 22. Use undetermined Coeff. Method to solve for $y(x)$ for y_p you may use substitution or Laplace as in class:

(i) $y^{(2)} + 14y' + 49y = x^7e^{-7x}$ [for y_p here Laplace method take much less time than substitution / I think!, note that here if you want use substitution $y_p = x^2(a_7x^7 + a_6x^6 + ... + a_0)$]

(ii) $y^{(2)} + 9y' + 8y = 2x + 3$ [here y_p will take same time using either method, note $y_p = ax + b$ find a, b]

(iii) $y^{(4)} + y^{(2)} = cos(x)$ [note $y_h = c_1 + c_2x + c_3cos(x) + c_4sin(x)$. Since cos(x) appears only once in y_h and the given LDE = cos(x), we conclude $y_p = x(acos(x) + bsin(x))$, I guess substitution here is easier to find y_p.]

(iv) $y^{(2)} + 6y' + 25y = e^{-3x}$[Note $y_h = e^{-3x}(c_1cos(4x) + c_2sin(4x))$, for y_p note that e^{-3x} is not a solution to y_h. Here, $y_p = ae^{-3x}$, find a. Both method will take the same time.]

(v) $y^{(2)} + 4y = sin(3x)$[Hint $y_h = c_1cos(2x) + c_2sin(2x)$, and $y_p = acos(3x) + bsin(3x)$. I guess substitution is easier here!]

(vi) $y^{(2)} + 4y' + 4y = (x^2 + 3x - 2)e^{-2x}$. [for y_p, Laplace much easier!!!]

(vii) Solve $x^4y^{(2)} + 5x^3y' + 3x^2y = 0$.

(viii) Solve $3y^{(2)} + 6y' + 3 = e^{-x}U(x - 1)$. [Note here for y_p must use laplace, I have no idea for substitution!]

QUESTION 23. Find the largest interval around x so that the LDE: $\frac{x-3}{\sqrt{3x+6}})y^{(4)} + (x-1)y = x^2 + 13, y'(1) = 7, y(1) = -6$ has a unique solution.

QUESTION 24. 1) Find $\ell\{e^{4x}\}$,
2)$\ell^{-1}\{\frac{s+3}{s^2-7s+6}\}$.

QUESTION 25. Solve the following DE:
1) $y^{(2)} + 8y' + 12y = 4$, $y(0) = 1$, $y'(0) = 0$.
2) $2y^{(2)} + 3y' + y = 0$, $y(0) = 3$ and $y'(0) = -2$

QUESTION 26. (i) A body at a temperature 50 F is placed outside where the temperature is 100 F. If after 5 minutes, the temperature of the body is 60 F. a) How long it will take the body to reach 60 F? b) What is the temperature of the body after 20 minutes?[Solution: $dT/dt = k(T - 100)$. Solve it by separable or first order linear. $T = ce^{-kt} + 100$. Note that $T(0) = 50$ and $T(5) = 60$. For (a): the answer is t = 15.4 minutes. For (b) the answer is 79.5F

(ii) Electric source of an electric circuit is given as $E(t) = 100(sin(t)+cos(t))$, the resistor-constance $R = 100$ Ohms, the capacitor-constant $c = 0.01$ Farad (No inductor is attached to the circuit). Initially, the charge on the capacitor is 2. Find the current $i(t)$ in the circuit at any time t. Find the steady-state-current.

[Solution: $i(t) = q'(t)$. We have $100q' + q(t)/0.01 = 100(sin(t) + cos(t))$. Hence $q'(t) + g(t) = sin(t) + cos(t)$. Hence $q(t) = \int e^t(sin(t) + cos(t))\,dt/e^t$. Thus $q(t) = ce^{-t} + sin(t)e^t$. Since $q(0) = 2$, we have $c = 2$. Thus $i(t) = q'(t) = -2e^{-t} + cos(t)e^t + sin(t)e^t$. Note that $-2e^{-t}$ reaches 0 when t is very huge. Hence the steady-state-current is determined by $cos(t)e^t + sin(t)e^t$]

(iii) Electric source of an electric circuit is given as $E(t) = 10t^2 + t$, the resistor-constance $R = 10$ Ohms, the inductor-constant $L = 0.5$ Henry (No capacitor is attached to the circuit). Initially, the current is 6 amperes. Find the current $i(t)$ in the circuit at any time t. Find the steady-state-current.

[Solution: $0.5i'(t) + 10i(t) = 10t^2 + t$. Thus $i'(t) + 20i(t) = 20t^2 + 2t$. Thus $i(t) = \int(20t^2 + 2t)e^{20t}\,dt/e^{20t}$. Hence $i(t) = t^2e^{20t} + ce^{-20t}$. Since $i(0) = 6$, $c = 6$. Thus $i(t) = t^2e^{20t} + 6e^{-20t}$. Since $6e^{-20t}$ approaches zero when t is very huge, the steady-state-current is t^2e^{20t}.]

(iv) Electric source of an electric circuit is given as $E(t) = 10sin(t)$, the resistor-constance $R = 180$ Ohms, the inductor-constant $L = 20$ Henry, capacitor-constant $c = 1/280$. Initially, the current is 1 ampere and no charge on the capacitor. Find the current $i(t)$ in the circuit at any time t. Find the steady-state-current.

[Solution: $20q^{(2)} + 180q' + 280q = 10sin(t)$. Thus **(1)** $q^{(2)} + 9q' + 14q = 0.5sin(t)$. We know $q(t) = q_h(t) + q_p(t)$. To find $q_h(t)$: set $q^{(2)} + 9q' + 14q = 0$. Thus $m^2 + 9m + 14 = 0$. $q_h(t) = c_1e^{-2t} + c_2e^{-7t}$. To find $q_p(t)$: we may use undetermined coefficient method: so $q_p(t) = asin(t) + bcos(t)$. by substitution in (1) we have $a = 13/500$ and $b = -9/500$. Thus $q(t) = c_1e^{-2t} + c_2e^{-7t} + \frac{13}{500}sin(t) + \frac{-9}{500}cos(t)$. Since $i(0) = q'(0) = 1$ and $q(0) = 0$. We have $c_1 = 110/500$ and $c_2 = -101/500$. Now derive $q(t)$ in order to get $i(t)$. Note that the steady-state-current is $\frac{13}{500}cos(t) + \frac{9}{500}sin(t)$]

QUESTION 27. (i) Find the critical points of $y' = -y^2 + 7y - 10$ and classify each as stable, semi-stable, unstable..

(ii) For the equation above, roughly sketch the solution graph if $y(0) = 4$.

(iii) Find the actual solution for the equation above(in I) (i.e. solve the given D. E), [Hint: use separable method, you need to use integration by fraction]]

(iv) A body at a temperature 50 F is placed outside where the temperature is 100 F. If after 5 minutes, the temperature of the body is 60 F. a) How long it will take the body to reach 60 F? b) What is the temperature of the body after 20 minutes?[Solution: $dT/dt = k(T - 100)$. Solve it by separable or first order linear. $T = ce^{-kt} + 100$. Note that $T(0) = 50$ and $T(5) = 60$. For (a): the answer is t = 15.4 minutes. For (b) the answer is 79.5F

(v) Let $A(t)$ be the amount of salt at any time t. A tank initially holds 100 gallons of a mixture containing 20 kg of salt. A fresh water is poured into the tank at the rate 5 gal/min, while the well stirred mixture leaves the tank at the same rate. a) Find A(t). [Hint $dA/dt = 0 - [5(1/100)]A(t) = 0 - A(t)/20$. So $A(t) = ce^{-t/20}$. Note that $A(0) = 20$ so find c, note that fresh water means each gallon enters the tank has 0 kg of salt!, concentration of salt in each gallon leaves the tank is $A(t)/(100 + 5t - 5t) = A(t)/100$]

(vi) Let $A(t)$ be the amount of salt at any time t. A 50-gal tank initially holds 10 gallons of fresh water (i.e. $A(0) = 0$). A mixture containing 1 kg of salt per gallon is poured into the tank at the rate 4 gal/min, while the well stirred mixture leaves the tank at rate 2 gal/min. a) Find A(t). b) Find the amount of salt at the moment of overflow? [Hint $dA/dt = 4 - [2(1/(10 + 4t - 2t)]A(t) = 4 - 2/(10 + 2t)A(t)$. Solve it using first order linear method, $A(t) = (40t + 4t^2)/(10 + 2t)$. Note that $A(0) = 0$, concentration of salt in each gallon leaves the tank is $A(t)/(10 + 4t - 2t) = A(t)/(10 + 2t)$. For b) note overflow occurs when volume of mixture in the tank = volume of the tank. Volume of the tank is 50. Volume of mixture (liquid) in the tank is (10 + 4t - 2t) = 10 + 2t). Thus set 10 + 2t = 50. We get t = 20. Now find $A(20)$, I guess A(20) = 48kg.]

QUESTION 28. (i) Find $\ell\{U(x-3)e^{2x}\}$

(ii) Find $\ell\{\int_0^x e^{-3r} sin(2r)\ dr\}$ [Hint : Note $e^{-3r} = e^{-3x}e^{3x-3r}$. Thus $\ell\{\int_0^x e^{-3r} sin(2r)\ dr\}$ = $\ell\{e^{-3x}\int_0^x e^{3x-3r} sin(2r)\ dr\}$]

(iii) Find $\ell^{-1}\{\frac{s+10}{(s+4)^4}\}$

(iv) Find $\ell^{-1}\{\frac{se^{-s}}{(s+3)^2+4}\}$

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Solve $y^{(2)} - 6y' + 8y = k(x)$, y(0) = $y'(0) = 0$ [hint : first write K(x) in terms of Unit function]

(ix) Solve : $y'(x) - \int_0^x 2e^{x-r}y(r)\ dr = xe^x$, $y(0) = 0$

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: `abadawi@aus.edu, www.ayman-badawi.com`